Exercise 6

Solve the equation $\sqrt{1-x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

Solution

Start by rewriting the PDE as

$$u_x + \frac{1}{\sqrt{1-x^2}}u_y = 0$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dy}{dx} = \frac{1}{1-x^2}, \quad y(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Integrating (1) gives

$$y = \sin^{-1} x + \xi.$$

Solving for ξ gives

$$\xi = y - \sin^{-1} x.$$

Therefore,

$$u(x, y) = f\left(y - \sin^{-1} x\right).$$

We're told that $u(0, y) = y$, though, so we can determine this unknown function, f.

$$u(0, y) = f(y) = y$$

This implies that $f(w) = w$, where w is any expression. Thus,

$$u(x, y) = y - \sin^{-1} x.$$

We can check that this is the solution of the PDE.

$$u_x = -\frac{1}{\sqrt{1-x^2}}$$
$$u_y = 1$$

$\sqrt{1-x^2}u_x + u_y = 0$, so this is the correct solution. The function is shown below in Figure 1. Shown below that in Figure 2 are the characteristic curves in the xy-plane for various values of ξ along with the line $x = 0$ (where the boundary condition is defined). Note that because $x = 0$ intersects each of the characteristics exactly once, the solution we obtained for $u(x, y)$ is valid for $-1 < x < 1$ and all y. It's because of the $\sqrt{1-x^2}$ term in the PDE that x is restricted.

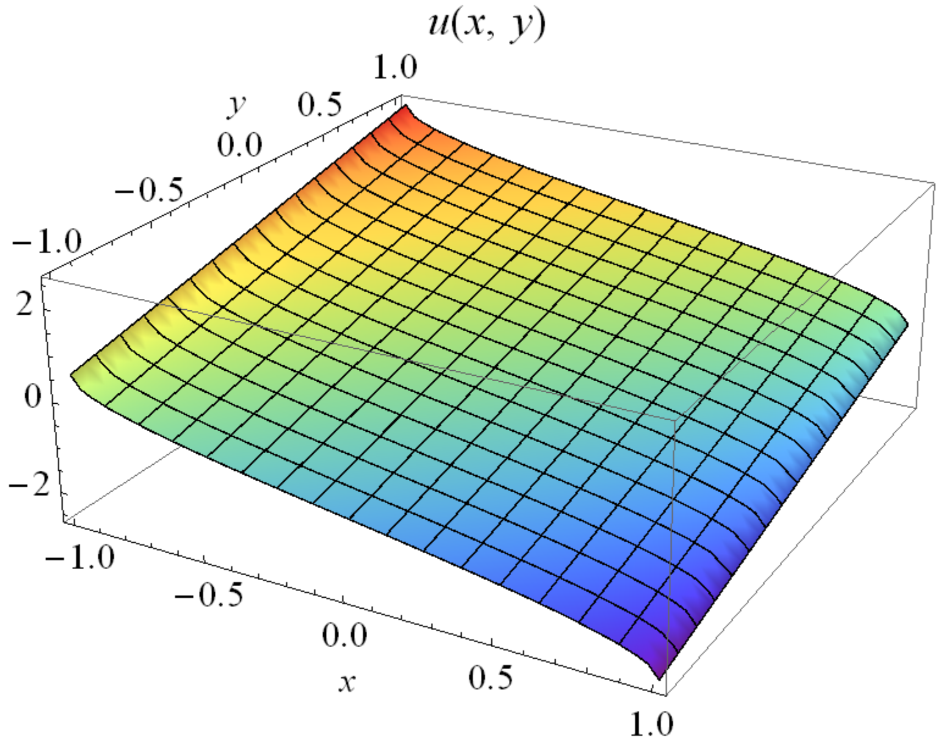

Figure 1: Plot of $u(x, y)$ for $-1 < x < 1$ and $-1 < y < 1$.

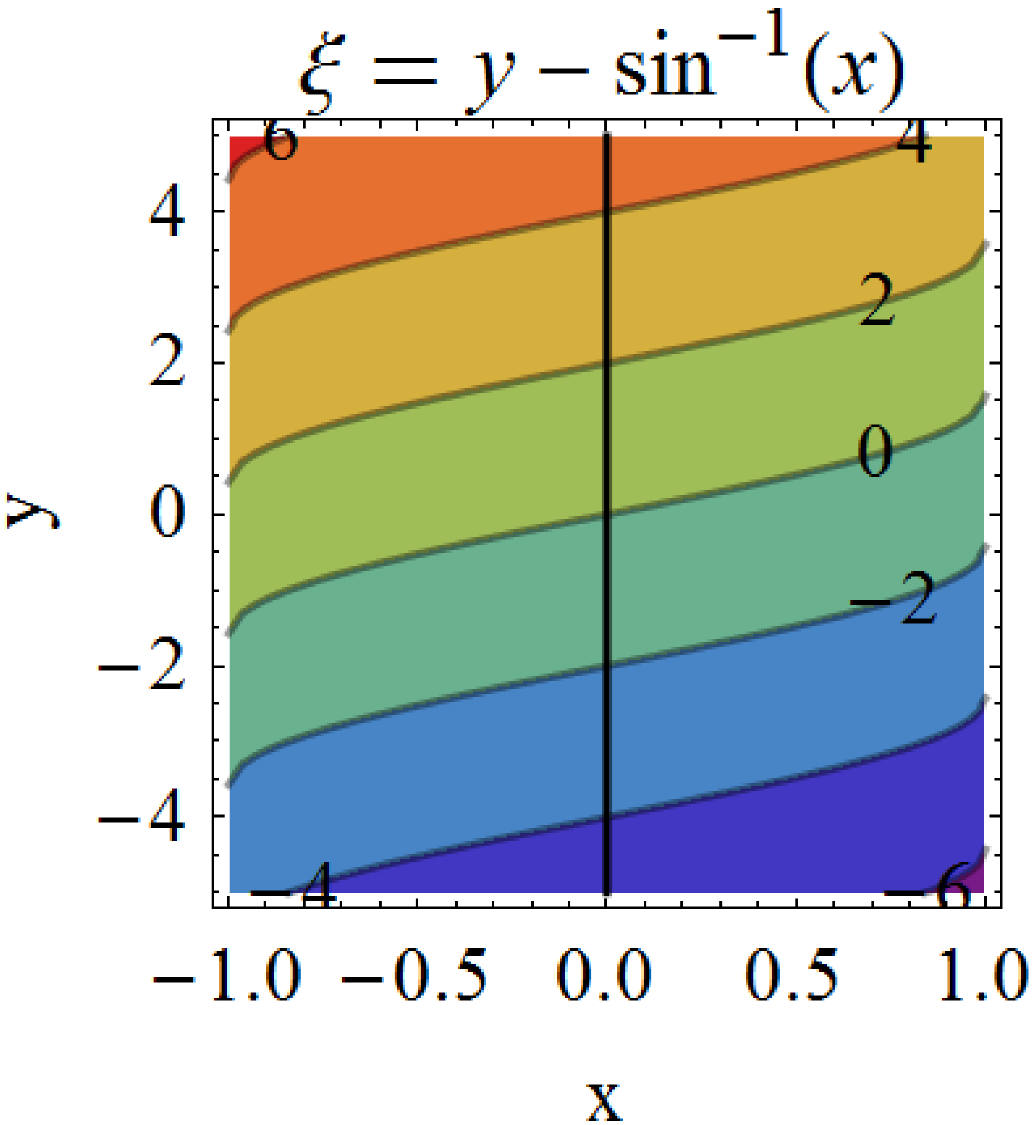

Figure 2: Plot of the characteristic curves along with the data curve in the xy-plane.